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TECHNICAL REPORT ARCCB-TR-88024

**A POSTERIORI ERROR ESTIMATION
OF ADAPTIVE FINITE DIFFERENCE
SCHEMES FOR HYPERBOLIC SYSTEMS**

DAVID C. ARNEY

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**US ARMY ARMAMENT RESEARCH,
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CLOSE COMBAT ARMAMENTS CENTER
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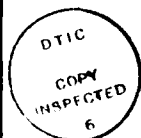
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difference scheme. We also discuss automatic time step selection procedures and the effects of artificial viscosity. Extrapolation schemes that produce separate estimates of the temporal and spatial discretization errors are presented and we show how these may be used to control local mesh refinement procedures. Several examples illustrating these procedures are presented.



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INTRODUCTION

With the use of adaptive methods to solve time-dependent partial differential equations, there exists a requirement to compute solutions on moving nonuniform grids. There is also a requirement to estimate the local discretization error as feedback to modify or refine the mesh. In this report, we discuss the MacCormack finite difference scheme and a Richardson extrapolation-based error estimation procedure that was used in the adaptive algorithm of Arney (ref 1) and Arney and Flaherty (refs 2,3) to solve time-dependent hyperbolic systems in one- and two-space dimensions. Examples of other adaptive methods with these requirements are Rai and Anderson (ref 4), Adjerdid and Flaherty (ref 5), Bell and Shubin (ref 6), and Davis and Flaherty (ref 7).

Finite difference methods use a mapping to transform the time and space variables from a moving nonuniform mesh to a stationary uniform mesh. The method used to compute the metrics of this transformation must be carefully chosen in order to preserve the stability, conservation, and accuracy of the scheme (cf. Thomas and Lombard (refs 8,9) and Hindman (refs 10,11)).

The MacCormack finite difference scheme has had wide use in solving Eulerian conservation laws for fluid dynamics. The recent use of artificial viscosity to make this scheme total variation diminishing (TVD) makes it more attractive as a general solver for problems with discontinuities (cf. Davis (ref 12) and Roe (13)). In the following section, we discuss the MacCormack scheme, our implementation of the differencing of the metric terms, adaptive selection of the time step, and the TVD artificial viscosity of Davis (ref 12). The Richardson's extrapolation-based error estimation method produces a pointwise approximation of the local discretization error which can be used to construct

References are listed at the end of this report.

several global measures of the discretization error. Next, we discuss our error estimate and its implementation on a moving mesh. Then we present computational results of solutions of hyperbolic problems. Computations were performed in one- and two-dimensions on stationary uniform and moving nonuniform grids. Lastly, we discuss the utility of our methods, the computational results, and future work.

SOLUTION SCHEME

Consider the hyperbolic vector systems of conservation laws in two-space dimensions

$$\vec{u}_t + \vec{f}_x(x,y,\vec{u},t) + \vec{g}_y(x,y,\vec{u},t) = 0 \quad , \quad (x,y) \in D, \quad t > 0 \quad (1)$$

$$\vec{u}(x,y,0) = \vec{u}_0(x,y), \quad (x,y) \in D \cup \partial D, \quad (2)$$

with appropriate well-posed conditions on the boundary ∂D of a rectangular domain D .

We chose to implement the MacCormack finite difference scheme for hyperbolic problems because of its general applicability. The MacCormack scheme, like most higher-order methods, will suffer a reduction in order on a moving nonuniform grid. Despite this fact, proper mesh moving and node placement by an effective adaptive procedure provide enough efficiency and accuracy to compensate for this order reduction.

MacCormack Scheme

In order to discretize Eq. (1), we introduce a transformation

$$\xi = \xi(x,y,t), \quad \eta = \eta(x,y,t), \quad \tau = t, \quad (3)$$

from the physical (x,y,t) domain to a computational (ξ,η,τ) domain where a uniform rectangular grid will be used. Under this transformation, Eq. (1) becomes

$$\vec{u}_\tau + \vec{u}_\xi \xi_t + \vec{u}_\eta \eta_t + \vec{f}_\xi \xi_x + \vec{f}_\eta \eta_x + \vec{g}_\xi \xi_y + \vec{g}_\eta \eta_y = 0 \quad (4)$$

The transformation metrics $(\xi_x, \xi_y, \xi_t, \eta_x, \eta_y, \eta_t)$ are related to the metrics $(x_\xi, x_\eta, x_\tau, y_\xi, y_\eta, y_\tau)$ by the identities

$$\begin{aligned} \xi_x &= \frac{y_\eta}{J}, \quad \xi_y = \frac{-x_\eta}{J}, \quad \xi_t = \frac{(y_\tau x_\eta - x_\tau y_\eta)}{J}, \quad \eta_x = \frac{-y_\xi}{J}, \\ \eta_y &= \frac{x_\xi}{J}, \quad \eta_t = \frac{(y_\xi x_\tau - x_\xi y_\tau)}{J}, \quad J = y_\eta x_\xi - x_\eta y_\xi \end{aligned} \quad (5)$$

Using Eq. (5) in Eq. (4) gives

$$\vec{u}_\tau + \vec{u}_\xi \frac{(y_\tau x_\eta - x_\tau x_\eta)}{J} + \vec{u}_\eta \frac{(y_\xi x_\tau - x_\xi y_\tau)}{J} + \vec{f}_\xi \frac{y_\eta}{J} + \vec{f}_\eta \left(\frac{-y_\xi}{J} \right) + \vec{g}_\xi \left(\frac{-x_\eta}{J} \right) + \vec{g} \frac{x_\xi}{J} = 0 \quad (6)$$

This equation can be rewritten in another form in the original transformation metrics by further substitutions of Eq. (5) into Eq. (2) as

$$\vec{u}_\tau - \vec{u}_\xi (x_\tau \xi_x + y_\tau \xi_y) - \vec{u}_\eta (x_\tau \eta_x + y_\tau \eta_y) + \vec{f}_\xi \xi_x + \vec{f}_\eta \eta_x + \vec{g}_\xi \xi_y + \vec{g}_\eta \eta_y = 0 \quad (7)$$

Some authors (cf. Hyman (ref 14) and Thompson (ref 15) prefer to write this equation in still another form as

$$\vec{u}_\tau + \vec{u}_\xi \xi_t + \vec{u}_\eta \eta_t + \vec{f}_\xi \xi_x + \vec{g}_\xi \xi_y + \vec{g}_\eta \eta_y = 0 \quad (8)$$

A uniform space-time grid having mesh spacing $\Delta\xi \times \Delta\eta \times \Delta\tau$ is introduced onto the computational domain. The finite difference solution at $(l\Delta\xi, m\Delta\eta, n\Delta\tau)$ is referred to as $\vec{u}_{l,m}^n$. A similar notation is used for the fluxes \vec{f} and \vec{g} and the metrics in Eq. (5). The two-step MacCormack scheme (ref 16) uses first-order forward temporal and spatial difference approximations in the predictor step, and first-order backward differences in the corrector step. The predicted solution $\vec{u}_{l,m}^{n+1}$ satisfies

$$\begin{aligned} \vec{u}_{l,m}^{n+1} &= \vec{u}_{l,m}^n - \frac{\Delta\tau}{\Delta\xi} [(\vec{u}_{l+1,m}^n - \vec{u}_{l,m}^n)(\xi_t)_{l,m}^n + (\vec{f}_{l+1,m}^n - \vec{f}_{l,m}^n)(\xi_x)_{l,m}^n \\ &\quad + (\vec{g}_{l+1,m}^n - \vec{g}_{l,m}^n)(\xi_y)_{l,m}^n] - \frac{\Delta\tau}{\Delta\eta} [(\vec{u}_{l,m+1}^n - \vec{u}_{l,m}^n)(\eta_t)_{l,m}^n \\ &\quad + (\vec{f}_{l,m+1}^n - \vec{f}_{l,m}^n)(\eta_x)_{l,m}^n + (\vec{g}_{l,m+1}^n - \vec{g}_{l,m}^n)(\eta_y)_{l,m}^n] \end{aligned} \quad (9)$$

The metrics $(\xi_x)_{l,m}^n$, etc. are computed by forward differences. The corrected solution $u_{l,m}^{n+1}$ satisfies

$$\begin{aligned} u_{l,m}^{n+1} = \frac{1}{2} \{ & u_{l,m}^n + \underline{u}_{l,m}^{n+1} - \frac{\Delta \tau}{\Delta \xi} [(\underline{u}_{l,m} - \underline{u}_{l-1,m}) (\xi_t)_{l,m}^{n+1} + (\underline{f}_{l,m} - \underline{f}_{l-1,m}) (\xi_x)_{l,m}^{n+1} \\ & + (\underline{g}_{l,m} - \underline{g}_{l-1,m}) (\xi_y)_{l,m}^{n+1}] - \frac{\Delta \tau}{\Delta \eta} [(\underline{u}_{l,m} - \underline{u}_{l,m-1}) (\eta_t)_{l,m}^{n+1} \\ & + (\underline{f}_{l,m} - \underline{f}_{l,m-1}) (\eta_x)_{l,m}^{n+1} + (\underline{g}_{l,m} - \underline{g}_{l,m-1}) (\eta_y)_{l,m}^{n+1}] \} \end{aligned} \quad (10)$$

with metrics computed by backward differences. The notation $\underline{f}_{l,m}^{n+1}$ denotes $f(\underline{u}_{l,m}^{n+1})$, etc. The use of first forward and backward difference approximations for the metrics implies that the transformation from the computational to the physical domain is piecewise trilinear in space and time for the predictor and corrector steps. Such low order difference approximations are responsible for reducing the orders of the MacCormack scheme. A smoother transformation and the use of higher-order difference approximations of the metrics could be used to maintain second order accuracy.

It was shown by Hindman (refs 10,11) that this differencing of Eq. (6) produces consistent approximations. Therefore, a uniform flow solution is maintained. Other conservative forms for the transformed equations were investigated by Hindman (ref 10) and found to be less efficient or needing special differencing of the metrics for computing consistent approximations.

Equation (4) is conservative on a moving mesh. We show this for a one-dimensional scalar conservation law by investigating the Rankine-Hugoniot jump conditions across a shock discontinuity. Consider a conservation law in the form

$$\frac{\partial}{\partial t} \left(\int_{-\infty}^{\infty} u \, dx \right) + f(u) \Big|_{-\infty}^{\infty} = 0 \quad (11)$$

The jump conditions for a discontinuity at $x = s(t)$ satisfy

$$\dot{s} = \frac{[f]}{[u]} \quad (12)$$

where $[q]$ indicates the jump in q , and $\dot{s} = \frac{ds}{dt}$ denotes the shock velocity (ref 17).

A conservation law on a moving mesh produced by a transformation of variables to a uniform stationary mesh satisfies

$$\left(\frac{\partial}{\partial \tau} + \xi_t \frac{\partial}{\partial \xi} \right) \int_{-\infty}^{\infty} u x_{\xi} d\xi + f(u) \Big|_{-\infty}^{\infty} = 0 \quad (13)$$

Assuming the existence of a shock discontinuity, $\xi = r(\tau)$ gives

$$\dot{r} x_{\xi}[u] + \int_{-\infty}^r (u x_{\xi})_{\tau} d\xi + \int_r^{\infty} (u x_{\xi})_{\tau} d\xi + f(u) \Big|_{-\infty}^{\infty} = 0 \quad (14)$$

Using the chain rule provides an integrable form

$$\dot{r} x_{\xi}[u] + \int_{-\infty}^r (u x_{\tau} - f)_{\xi} d\xi + \int_r^{\infty} (u x_{\tau} - f)_{\xi} d\xi + f(u) \Big|_{-\infty}^{\infty} = 0 \quad (15)$$

Integration of this equation gives jump conditions in the computational domain as

$$\dot{r} x_{\xi}[u] - [f] + x_{\tau}[u] = 0 \quad (16)$$

Since $s(t)$ and $r(\tau)$ are related by

$$\dot{s} = \dot{r} x_{\xi} + x_{\tau} \quad (17)$$

the appropriate jump condition in Eq. (12) is recovered.

Variable Time Step

The explicit MacGormack scheme has a stability restriction that limits the time step allowed for a given spatial mesh. For efficient computation, the time step should be adaptively set close to the maximum allowed by the Courant, Friedrichs, Lewy theorem (ref 18)

$$\Delta\tau = \frac{0.8}{2\sqrt{2} \max(\psi, \omega)} \quad (18)$$

The computational mesh has been selected to have spacing $\Delta\xi = \Delta\eta = 1$ and the constant 0.8 provides a 20 percent margin of safety. The quantities ψ and ω are the spectral radii of one-dimensional conservation laws on moving meshes, i.e.,

$$\psi = \max[(\lambda_i - x_\tau)\xi_x + (\rho_i - y_\tau)\xi_y] \quad (19a)$$

$$\omega = \max[(\lambda_i - x_\tau)\eta_x + (\rho_i - y_\tau)\eta_y] \quad (19b)$$

where λ_i and ρ_i are eigenvalues of $\vec{f}_u(\vec{u})$ and $\vec{g}_u(\vec{u})$. These eigenvalues and the metrics in Eqs. (19a) and (19b) are evaluated at the beginning of each time step.

Artificial Viscosity

The MacCormack scheme, being a second order accurate centered scheme, produces spurious oscillations near discontinuities. In order to eliminate or reduce these oscillations, artificial viscosity or dissipation is added to the solution to diffuse the discontinuity. The viscosity is often problem-dependent, and considerable "fine tuning" is usually needed to balance the effects of the spurious oscillations and diffusion (ref 19).

We use an artificial viscosity model due to Davis (ref 12) which is not problem-dependent and only requires knowledge of ψ and ω . This artificial viscosity model is designed to convert the MacCormack scheme into a total variation diminishing (TVD) scheme in one-dimension. A scheme is TVD if the total variation of the solution to an initial value problem is nonincreasing in time. Recent research efforts have resulted in the development of other second order accurate TVD schemes (cf. Osher and Chakravarthy (ref 20) and Warming and Beam (ref 21)).

The artificial viscosity of Davis (ref 12) is based on a flux limiter that does not depend on explicitly determining the upwind direction and, with a recent modification by Roe (ref 13), does not affect the region of stability of the MacCormack scheme. Because the MacCormack scheme also does not determine the upwind direction, the combined use of the MacCormack scheme and Davis' artificial viscosity is computationally simpler to perform than many other TVD schemes. The artificial viscosity terms are calculated from the solution data at the beginning of the time step. For two-dimensional problems, separate dissipative terms are calculated in the ξ and η directions, respectively.

ERROR ESTIMATION

Accurate a posteriori error estimation is an integral part of an adaptive software system. Error estimation can be the most expensive part of an adaptive procedure and an important goal is to find accurate and inexpensive ways of estimating the discretization error (cf. Babuska, et al. (refs 22,23)). The error estimation technique is dependent on many factors, including the type of solver used in the algorithm, the type of error to be determined, and the norm in which the error estimate is to be measured. It is most desirable to have a procedure that provides pointwise estimates of the error which can then be used to find estimates in several local and global norms.

Mesh nonuniformity affects the accuracy and convergence of numerical schemes and error estimation. The effects of the mesh on the solution scheme have been studied by Ciment (ref 24), Fritts (ref 25), Hoffman (ref 26), Osher and Sanders (ref 27), Sanders (ref 28), and Mastin (ref 29). Error analysis seems to be more natural and further developed for finite element schemes, especially for elliptic and parabolic problems (cf. Adjerid and Flaherty (refs 5, 30), Zienkiewicz et al. (refs 31,32), and Babuska and Rheinboldt (refs 33,34)),

where relatively inexpensive local calculations are used to provide accurate global spatial error estimates. More study needs to be done to find less expensive and more accurate error estimates for finite difference schemes for hyperbolic problems.

We calculate the local temporal and spatial portions of the discretization error, using an algorithm based on Richardson's extrapolation. Flaherty and Moore (refs 35,36) and Berger and Oliger (ref 37) also use Richardson's extrapolation to estimate error on uniform meshes for their local mesh refinement algorithms.

Richardson's Extrapolation Error Estimation

We develop the error estimation for the second order MacCormack scheme for a linear scalar problem in two dimensions. Separate pointwise estimates at a general spatial node i , at time t , for the local temporal error $E_i^t(t)$ and local spatial error $E_i^s(t)$ are obtained with two different extrapolation procedures.

Consider a uniform mesh with spacing $\Delta x \times \Delta y$ and time step Δt . Let the exact solution at node i and time t be denoted as $u_i(t)$, the numerical solution by the MacCormack scheme at the same point and time as $U_i(t; \Delta x, \Delta y, \Delta t)$, and the MacCormack finite difference operator as $L(\Delta x, \Delta y, \Delta t)$, i.e.,

$$U_i(t + \Delta t; \Delta x, \Delta y, \Delta t) = L(\Delta x, \Delta y, \Delta t)U_i(t; \Delta x, \Delta y, \Delta t) \quad (20)$$

Assume that the local error has a Taylor's series expansion of the form

$$u_i(t) - U_i(t; \Delta x, \Delta y, \Delta t) = \Delta t[c_1 \Delta t^2 + c_2 \Delta x^2 + c_3 \Delta y^2 + \dots] \quad (21)$$

where the constants c_1, c_2, c_3, \dots are independent of the mesh spacing.

To estimate the spatial component of the error, we calculate a solution on a mesh of double spatial size ($2\Delta x \times 2\Delta y$) with the same time step (Δt). The local error on this mesh satisfies

$$u_i(t + \Delta t) - U_i(t + \Delta t; 2\Delta x, 2\Delta y, \Delta t) = \Delta t[c_1 \Delta t^2 + 4c_2 \Delta x^2 + 4c_3 \Delta y^2 + \dots] \quad (22)$$

Subtracting Eq. (22) from Eq. (21) and neglecting higher order terms, we obtain an expression for the leading term in the spatial portion of the local error for the MacCormack scheme on the $\Delta x \times \Delta y \times \Delta t$ mesh as

$$\begin{aligned} E_i^s(t+\Delta t) &:= \Delta t [c_2 \Delta x^2 + c_3 \Delta y^2] \\ &= \frac{1}{3} [U_i(t+\Delta t; 2\Delta x, 2\Delta y, \Delta t) - U_i(t+\Delta t; \Delta x, \Delta y, \Delta t)] \end{aligned} \quad (23)$$

Similarly, an estimate of the temporal portion of the local error, $E_i^t(t+\Delta t)$, can be calculated by computing another solution on the $\Delta x \times \Delta y$ spatial mesh using two time steps of $\Delta t/2$, subtracting this result from Eq. (21), and retaining the leading order term as

$$\begin{aligned} E_i^t(t+\Delta t) &:= \Delta t [c_1 \Delta t^2] \\ &= \frac{4}{3} [U_i(t+\Delta t; \Delta x, \Delta y, 2(\frac{\Delta t}{2})) - U_i(t+\Delta t; \Delta x, \Delta y, \Delta t)] \end{aligned} \quad (24)$$

The leading term of the local error at node i and time $t+\Delta t$ is

$$E_i(t+\Delta t) = E_i^t(t+\Delta t) + E_i^s(t+\Delta t) \quad (25)$$

There are several disadvantages to this technique that should be noted: (1) the error cannot be calculated for nodes on or adjacent to the boundary; (2) the solution must be smooth enough for the c_1 , c_2 , and c_3 to exist; (3) the error estimation costs approximately three times more to compute than the solution; and (4) the mesh must be uniform. Equation (25) may still be useful as a mesh refinement or motion indicator even in situations where jumps in the solution render it invalid as an estimate of the error.

Richardson's extrapolation can be done in a more classic manner provided that we are willing to forego separate spatial and temporal error estimates. Thus, the error at node i in a solution on a mesh having spacing $\Delta x \times \Delta t$ is estimated by calculating a second solution on a mesh with spacing $\Delta x/2$ using two time steps of $\Delta t/2$. According to Eq. (22), the local error on this mesh

satisfies

$$u_i(t+\Delta t) - U_i(t+\Delta t; \Delta x/2, 2(\Delta t/2)) = \Delta t [c_1 \Delta t^2/4 + c_2 \Delta x^2/4 + \dots] \quad (26)$$

Subtracting Eq. (26) from Eq. (22) and neglecting higher order terms, we can obtain error estimates for either $U_i(t+\Delta t; \Delta x, \Delta t)$ or $U_i(t+\Delta t; \Delta x/2, 2(\Delta t/2))$ provided that node i is common to both meshes. Our adaptive method carries the fine grid solution forward in time; thus, we estimate its error as

$$\begin{aligned} E_i(t+\Delta t) &= \frac{1}{4} \Delta t (c_1 \Delta t^2 + c_2 \Delta x^2) \\ &= \frac{1}{3} [U_i(t+\Delta t; \frac{\Delta x}{2}, 2(\frac{\Delta t}{2})) - U_i(t+\Delta t; \Delta x, \Delta t)] \end{aligned} \quad (27)$$

Using this procedure, the error can now be calculated at nodes adjacent to boundaries. Even though this error estimate costs four times more to compute than the solution, we only incur this overhead in the first time step. No additional cost is incurred if portions of the mesh have to be refined because the solution on the refined mesh has already been computed and stored while estimating the error for the coarser parent mesh.

Error Estimation for a Moving Nonuniform Mesh

Nonuniformity of the mesh changes the discretization error of the MacCormack scheme. For simplicity, we will determine this error and analyze its effects on the Richardson extrapolation error estimation using a linear scalar problem in one-space dimension

$$\vec{u}_t + b \vec{u}_x = 0 \quad (28)$$

The local error for the MacCormack method on a one-dimensional moving nonuniform mesh is

$$\begin{aligned} u_i(t+\Delta t) - U_i(t+\Delta t; \Delta x, \Delta t) &= \Delta t [-\frac{b}{4} (\Delta x_r^{n+1} - \Delta x_l^n) u_{xx} \\ &\quad - \Delta t b^2 (1 - (\frac{\Delta x_r^{n+1}}{\Delta x_l^n})) u_{xx}] + c_1 \Delta t^2 + c_2 \Delta x^2 \end{aligned} \quad (29)$$

where Δx_l^n and Δx_r^n are the mesh sizes on the left and right node i at time step n , respectively, and $\Delta x = \max(\Delta x_r^{n+1}, \Delta x_l^{n+1})$. On the moving nonuniform mesh, both the temporal and spatial error components contain second order terms, whereas the error on a uniform mesh is third order. The previous analysis can be used to show that the leading component of the temporal error is

$$\begin{aligned} E_i^t(t+\Delta t) &:= \Delta t \left[-\Delta t b^2 \left(1 - \frac{\Delta x^{n+1}}{\Delta x_l^n} \right) u_{xx} \right] \\ &= 2[U_i(t+\Delta t; \Delta x, 2(\frac{\Delta t}{2})) - U_i(t+\Delta t; \Delta x, \Delta t)] \end{aligned} \quad (30)$$

Calculation of the spatial portion of the error is more difficult since the temporal portion of the error does not cancel upon subtraction of solutions calculated on two spatially different meshes. We overcome this difficulty and also greatly simplify the procedure in two dimensions by constraining the mesh to maintain double size increments for special nodes of the moving coarse mesh. This constrained grid structure consists of a coarse mesh, shown with darker lines in Figure 1, containing properly nested fine cells created by binary division of the sides of the coarse cells, shown by lighter lines in Figure 1. The vertices of the coarse cells are denoted as "independent moving nodes." Error estimates are calculated for these nodes. The remaining nodes in the mesh of Figure 1 are "dependent moving nodes" which must be moved to maintain the constrained grid structure. A solution is computed for these "dependent moving nodes," but no error estimate is obtained.

For the "independent moving nodes," the spatial error calculation can proceed as for a uniform mesh; therefore, the local spatial error estimate is

$$\begin{aligned} E_i^s(t+\Delta t) &= \Delta t \left[-\frac{b}{4} (\Delta x_r^{n+1} - \Delta x_l^n) u_{xx} \right] \\ &= U_i(t+\Delta t; \Delta x, \Delta t) - U_i(t+\Delta t; 2\Delta x, \Delta t) \end{aligned} \quad (31)$$

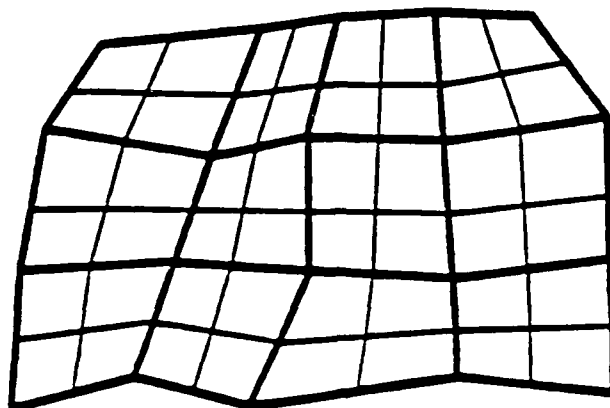


Figure 1. Spatial structure of the moving coarse mesh (bold lines) with embedded fine mesh (fine lines) used for the error estimation.

The above analysis extends directly to two dimensions; hence, we have a Richardson extrapolation-based procedure of estimating error on a moving nonuniform grid. In practice, we test the need for local uniformity and, if found, use formulas in Eqs. (23) through (25) to compute error estimates.

Error estimation for systems of equations involves the use of a vector norm at node i and time (t) . The examples in the following section use the maximum norm, i.e.,

$$E_i(t) := \max_{1 \leq j \leq N} |E_{i,j}(t)| \quad (32)$$

where N is the number of equations in the system and $E_{ij}(t)$ is the local error estimate for the j th component of the solution vector at node i .

COMPUTATIONAL EXAMPLES

The solution and local error estimation procedures are applied to four examples. In Example 1, we demonstrate the capability of the MacCormack scheme with Davis' TVD artificial viscosity on a moving nonuniform mesh. In Example 2, we investigate a one-dimensional problem using a modified form of the error

estimate in Eqs. (27) and (28). Examples 3 and 4 illustrate the performance of the error estimation procedure on a problem having a smooth solution and one with a jump in the first derivative, respectively. We investigate the accuracy and convergence of the local error estimator by determining an effectivity index

$$\theta = \frac{\|E\|_1}{\|e\|_1} \quad (33a)$$

at a fixed time t for several different meshes and different adaptive strategies. Here e and E are the exact and estimated errors, respectively. The L_1 norm,

$$\|E\|_1 := \iint E dx dy \quad (33b)$$

is obtained by assuming E to be a piecewise constant function.

Example 1

Consider the initial-boundary value problem

$$u_t - yu_x + xu_y = 0, \quad t > 0, \quad -1.2 \leq x \leq 1.2, \quad -1.2 \leq y \leq 1.2 \quad (34)$$

$$u(x, y, 0) = \begin{cases} 0 & , \text{ if } (x - \frac{1}{2})^2 + 1.5y^2 \geq \frac{1}{16} \\ 1 - 16((x - \frac{1}{2})^2 + 1.5y^2) & , \text{ otherwise} \end{cases} \quad (35)$$

and

$$u(1.2, y, t) = u(-1.2, y, t) = u(x, -1.2, t) = u(x, 1.2, t) = 0 \quad (36)$$

$$u(x, y, t) = \begin{cases} 0 & , \text{ if } C < 0 \\ C & , \text{ if } C \geq 0 \end{cases} \quad (37)$$

where

$$C = 1 - 16((x \cos t + y \sin t - \frac{1}{2})^2 + 1.5(y \cos t - x \sin t)^2) \quad (38)$$

Equations (37) and (38) represent a moving elliptical cone rotating counterclockwise around the origin with period 2π . This problem was proposed as a test problem by Gottlieb and Orszag (ref 38) and was used as a test problem in a survey by McRae et al. (ref 39).

We show the sequence of meshes that were generated at $t = 0, 1.6$, and 3.2 using the adaptive mesh moving method of Arney and Flaherty (ref 2) in Figures 2, 3, and 4, respectively. Arney and Flaherty's mesh moving method (ref 2) utilizes the error estimates (see Error Estimation section of this report) to concentrate the mesh in the high-error region beneath the cone and to follow it as it rotates. It also increases the accuracy of the solution and reduces oscillations in the wake following the cone. However, small oscillations are still present. Next, we solve this problem with the same moving mesh technique by using Davis' artificial viscosity (ref 12) with the MacCormack scheme. Surface and contour plots of solutions with and without artificial viscosity are shown in Figures 5 and 6. There is no artificial wake behind the cone when artificial viscosity is used. However, the artificial viscosity slightly diffuses the cone, widening its base and reducing its peak from 1.0 to 0.88 .

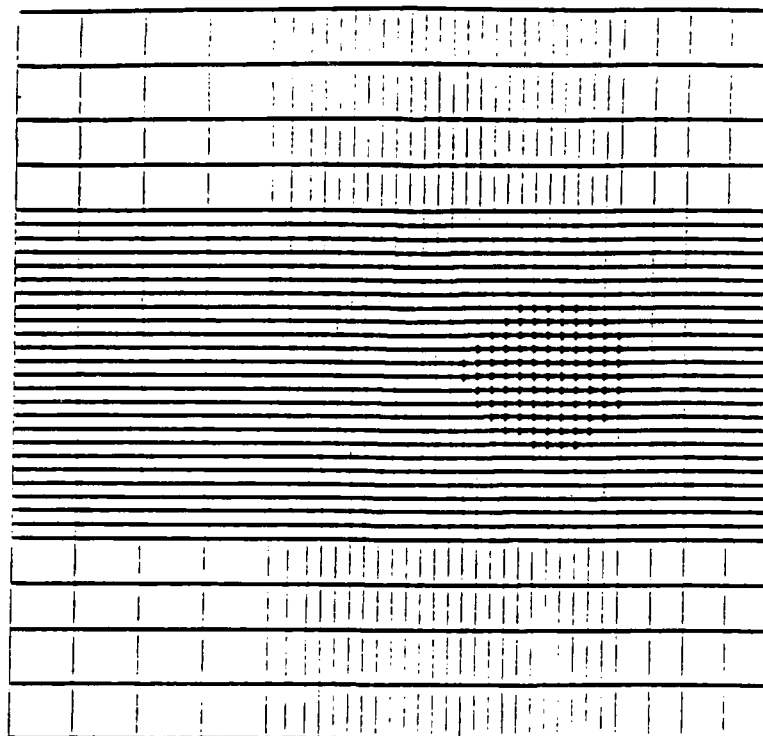


Figure 2. Initial mesh for Example 1.

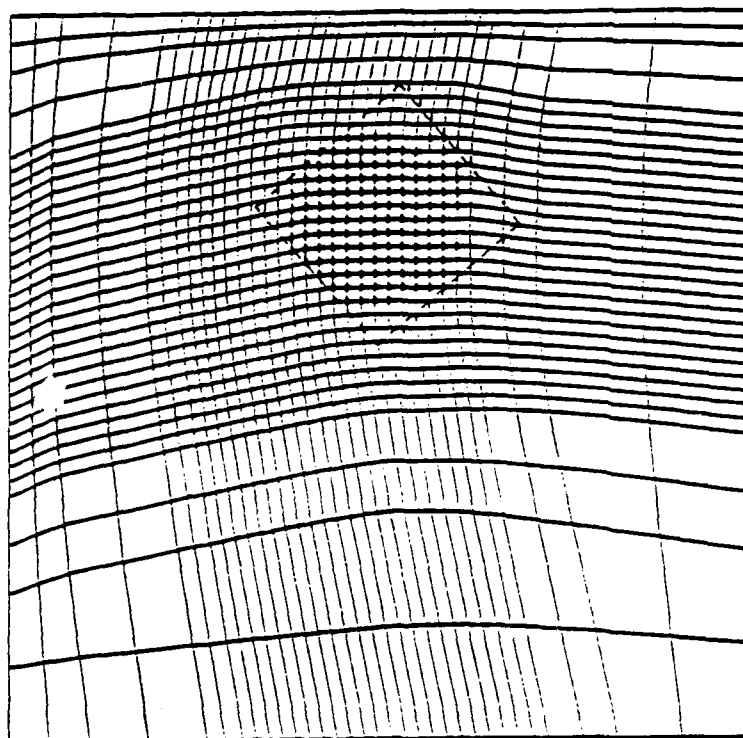


Figure 3. Mesh of Example 1 at $t = 1.6$. Nodes have moved with the rotating cone.

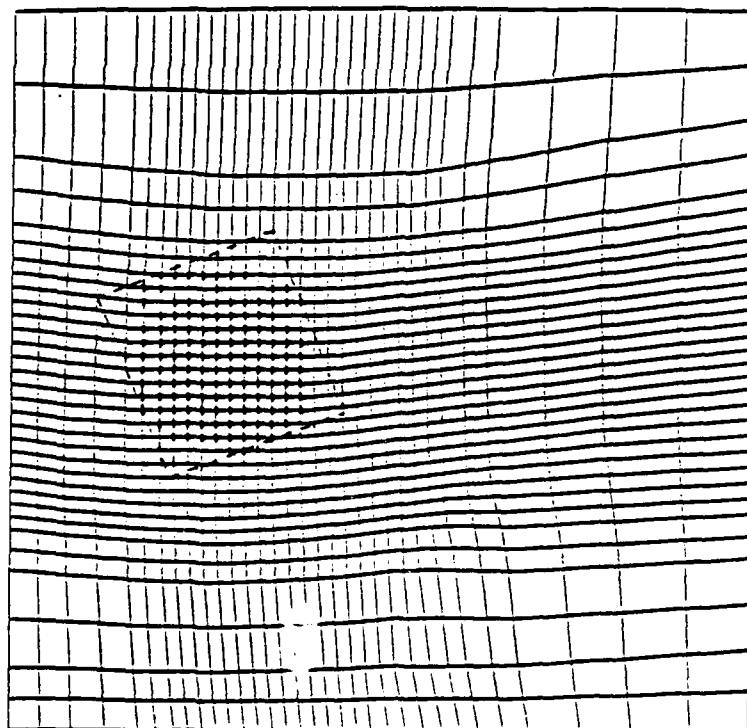


Figure 4. Mesh of Example 1 at $t = 3.2$. Nodes have moved with the cone for one-half rotation.

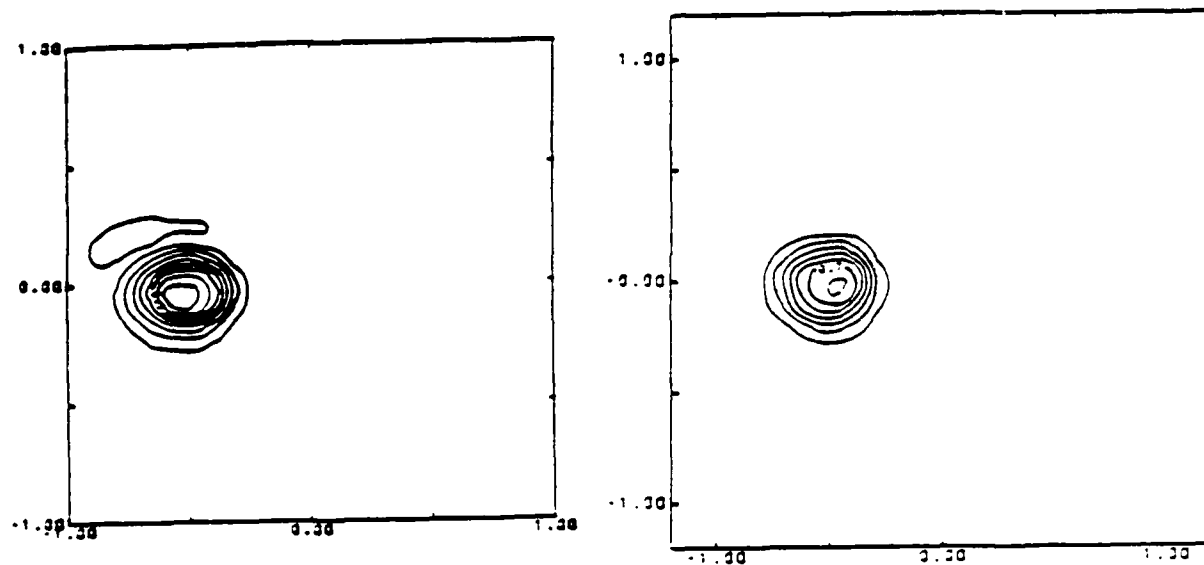


Figure 5. Contour plots of the solutions of Example 1 on a moving mesh without artificial viscosity (left) and with artificial viscosity (right) at $t = 3.2$.

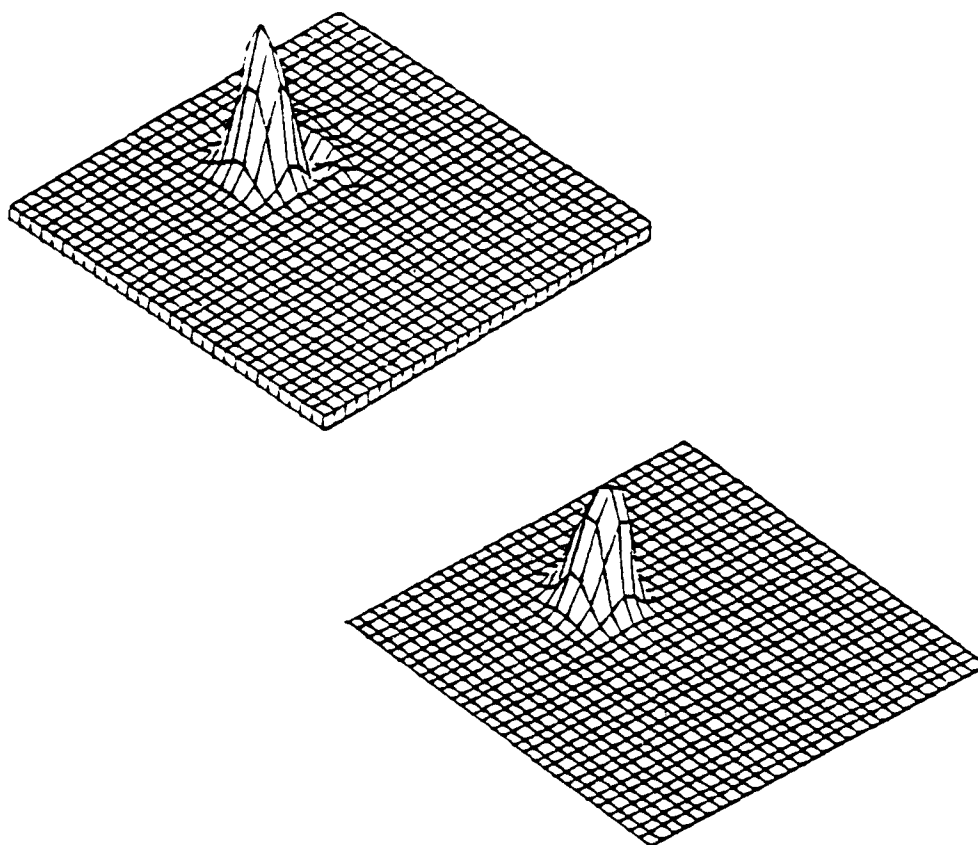


Figure 6. Surface plots of the solutions of Example 1 on a moving mesh without artificial viscosity (top) and with artificial viscosity (bottom) at $t = 3.2$.

Example 2

We consider an application of the direct Richardson's extrapolation error estimation procedure, Eq. (28), to the one-dimensional linear scalar equation

$$u_t + u_x = 0, \quad t > 0, \quad 0 < x < 0.8 \quad (39)$$

with initial and Dirichlet boundary conditions specified so that the exact solution is

$$u(x,t) = \frac{1}{2} [1 - \tanh 100 (x-t-0.2)] \quad (40)$$

This solution is a relatively steep wave that moves at unit speed across the domain.

We solved this problem for one time step on seven different uniform meshes having N computational cells per time step in order to investigate accuracy and convergence of the error estimate. Table I shows the results obtained from these calculations.

TABLE I. EXACT AND ESTIMATED ERRORS FOR DIFFERENT MESH SIZES FOR EXAMPLE 2

Δt	N	Exact Error $\ e\ _1$	Estimated Error $\ E\ _1$	Effectivity Ratio θ
0.1	8	0.447×10^{-2}	0.351×10^{-1}	7.60
0.05	16	0.234×10^{-2}	0.132×10^{-1}	5.55
0.025	32	0.106×10^{-2}	0.236×10^{-2}	2.22
0.0125	64	0.257×10^{-3}	0.138×10^{-3}	0.54
0.00625	128	0.294×10^{-4}	0.380×10^{-4}	1.29
0.00312	256	0.303×10^{-5}	0.453×10^{-5}	1.49
0.00156	512	0.538×10^{-6}	0.661×10^{-6}	1.23

We also solved this problem using Arney and Flaherty's adaptive local refinement procedure (ref 3) on a base mesh having $\Delta x = \Delta t = 0.1$ with a local error tolerance of $1/128$. The mesh created by the local refinement algorithm is shown in Figure 7 and the solutions computed at each base time step are shown in Figure 8.

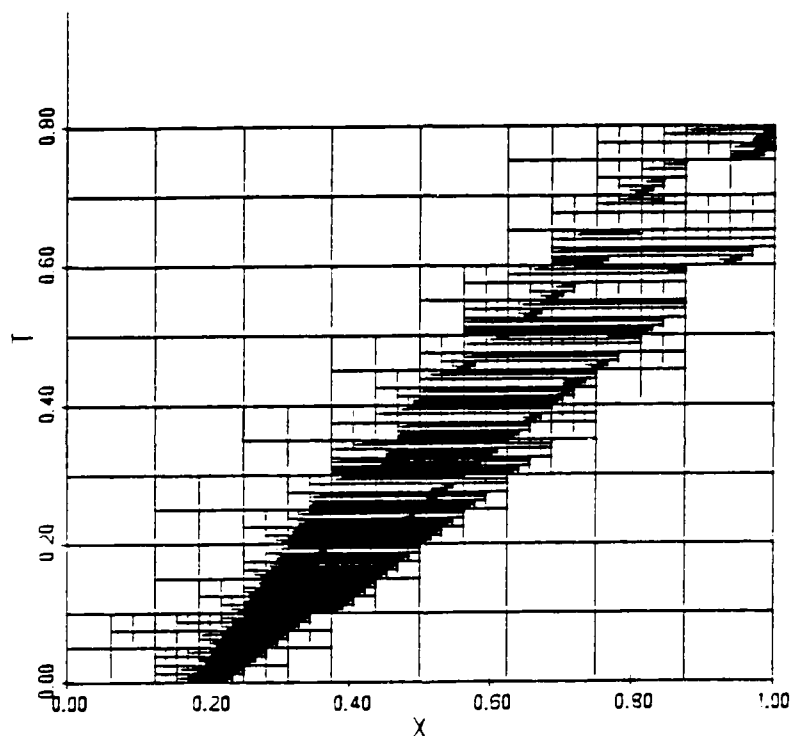


Figure 7. Adaptive mesh of Example 2.

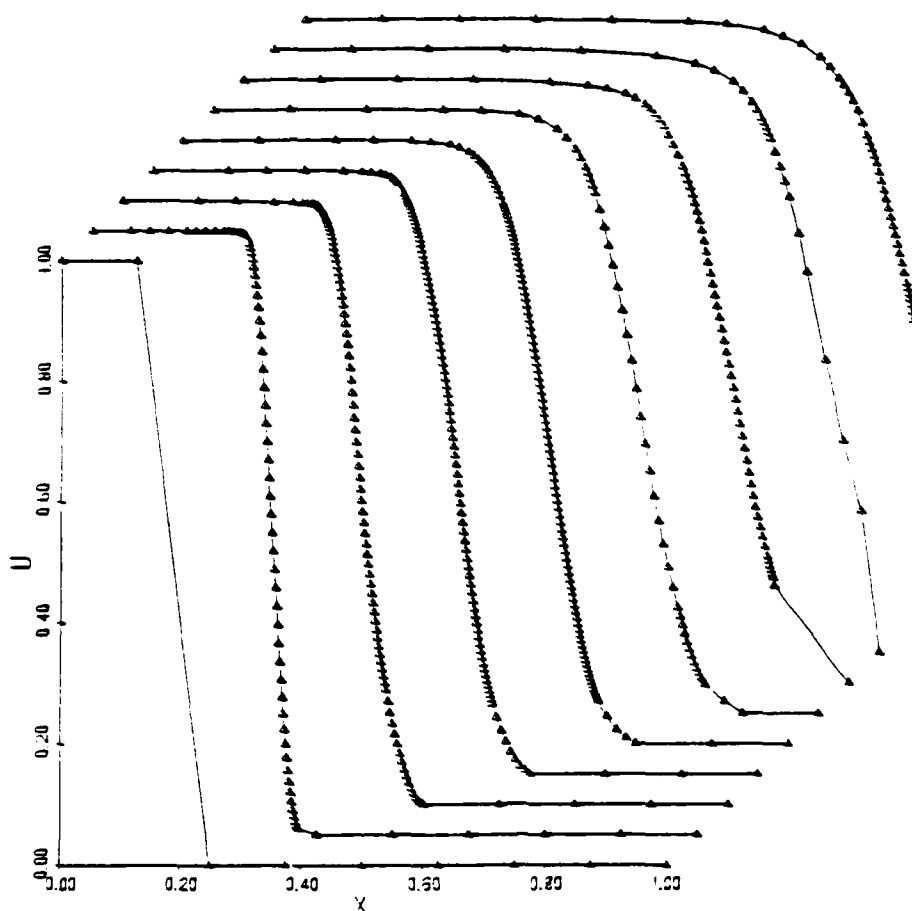


Figure 8. Solutions at base time steps for the adaptive local refinement procedure of Example 2.

The adaptive composite mesh of Figure 7 shows a distinct pattern associated with using the MacCormack scheme with our local refinement strategy. Spacious oscillations of the solution on the base mesh cause several levels of refinement which drastically reduce the base mesh spacing at the beginning of each base time step. However, once these oscillations have been controlled, the need for refinement is reduced at the later stages of the adaptive procedure. This situation could be alleviated by including an artificial viscosity model with the MacCormack scheme.

Example 3

Consider the linear scalar hyperbolic differential equation

$$u_t + 2u_x + 2u_y = 0, \quad t > 0, \quad 0.2 \leq x \leq 1.2, \quad 0 < y \leq 1 \quad (41)$$

with initial conditions

$$u(x,y,0) = \frac{(1 - \tanh 3(x - 0.1y + 0.1))}{2} \quad (42)$$

and with Dirichlet boundary conditions specified so that the exact solution of this problem is

$$u(x,y,t) = \frac{(1 - \tanh 3(x - 0.1y - 1.8t + 0.1))}{2} \quad (43)$$

This solution is a smooth wave that moves at an angle of 45 degrees across the domain. The problem was selected to show the convergence and accuracy of Richardson's extrapolation error estimation procedure (Eqs. (28) and (29)). We solve Eqs. (41) and (42) for one time step, $\Delta t = 0.012$, on eight different meshes. The mesh strategy of each calculation is described as follows:

1. a stationary uniform (10 x 10) rectangular mesh
2. a stationary uniform (20 x 20) rectangular mesh
3. a stationary uniform (40 x 40) rectangular mesh
4. a stationary uniform (60 x 60) rectangular mesh
5. a stationary (40 x 40) mesh of nonuniform quadrilateral cells
6. a moving (20 x 20) mesh with uniform rectangles
7. a moving (20 x 20) mesh of nonuniform quadrilateral cells
8. a moving (40 x 40) mesh of nonuniform quadrilateral cells

Table II shows the results from these calculations by comparing the exact errors and the effectivity indices for the eight strategies.

TABLE II. EXACT AND ESTIMATED ERRORS FOR DIFFERENT MESH STRATEGIES FOR EXAMPLE 3

Mesh Strategy (From Above)	Exact Error $\ e\ _1$	Estimated Error $\ E\ _1$	Effectivity Ratio θ
1	0.0111	0.0071	0.64
2	0.00370	0.00318	0.86
3	0.000942	0.000908	0.96
4	0.000367	0.000368	1.00
5	0.000399	0.000418	1.04
6	0.00136	0.00124	0.91
7	0.000411	0.000370	0.90
8	0.000167	0.000156	0.94

Strategies 1 through 4 show the convergence of the error estimates on uniform meshes as the number of nodes increases. These errors show a rate of convergence of $O(\Delta x^2, \Delta y^2)$ which is predicted in Eq. (21). Comparison of the errors of strategies 3 and 5 shows the error is cut in half by computing with a better nonuniform stationary mesh. Further comparison of strategies 5 and 7 shows another reduction of error by half when the mesh is properly moved. The nonuniformity of the mesh in strategies 5, 7, and 8 produces little change in the effectivity of the error estimation. These nonuniform mesh computations indicate a convergence rate $O(\Delta x^{1.32}, \Delta y^{1.32})$.

Example 4

Consider the linear scalar hyperbolic differential equation

$$u_t + u_x + 0.25u_y = 0, \quad t > 0, \quad 0.2 \leq x \leq 1.2, \quad 0 \leq y \leq 1 \quad (44)$$

with initial conditions

$$u(x,y,0) = \begin{cases} 0 & , \text{ if } y < -4x + 1.2 \\ 0.8 & , \text{ if } y > -4x + 1.6 \\ -8x - 2y + 3.2 & , \text{ otherwise,} \end{cases} \quad (45)$$

and with Dirichlet boundary conditions

$$u(x,y,0) = \begin{cases} 0 & , \text{ if } y - 0.25t < -4(x-t) + 1.2 \\ 0.8 & , \text{ if } y - 0.25t > -4(x-t) + 1.6 \\ -8(x-t) - 2(y-0.25t) + 3.2, & \text{ otherwise.} \end{cases} \quad (46)$$

The solution of this problem is an oblique ramplike wave front that moves at an angle of 14 degrees across the domain. The solution has a jump in its first partial derivatives at the top and bottom edges of the wave front. We expect some difficulty in estimating the error near locations where the derivatives jump. In the region of the front itself, the gradient of the solution is constant and there is no error in the solution or in the error estimate.

We solved this problem for one time step, $\Delta t = 0.015$, for the following six mesh strategies:

1. a stationary uniform (12 x 12) rectangular mesh
2. a stationary uniform (24 x 24) rectangular mesh
3. a stationary uniform (48 x 48) rectangular mesh
4. a stationary uniform (64 x 64) rectangular mesh
5. a stationary (24 x 24) mesh of nonuniform quadrilateral cells
6. a moving (24 x 24) mesh of nonuniform quadrilateral cells.

Table III shows the results of these strategies.

TABLE III. EXACT AND ESTIMATED ERRORS FOR DIFFERENT MESH STRATEGIES FOR EXAMPLE 4. THE ERROR ESTIMATE IS ACCURATE, BUT THE SOLUTION APPEARS TO BE CONVERGING

Mesh Strategy	Exact Error $\ e\ _1$	Estimated Error $\ E\ _1$	Effectivity Ratio θ
1	0.0058	0.0016	0.28
2	0.00275	0.00110	0.40
3	0.000866	0.000479	0.55
4	0.000400	0.000222	0.56
5	0.00144	0.00078	0.54
6	0.000720	0.000349	0.49

The results are once again as expected. The error estimate of this problem with a jump in the derivative is not as accurate as the smooth solution of Example 3. However, the error estimate still shows signs of converging to the exact error in L_1 for the uniform meshes of strategies 1 through 4. Once again, the better nodal placement of the initial mesh by the mesh generator of Arney (ref 1) reduces the error by half from a uniform mesh. Also, moving the mesh by the method of Arney and Flaherty (ref 2) reduces the error by half again.

CONCLUSION

We have shown that MacCormack's finite difference scheme and error estimation based on Richardson's extrapolation can be used on moving grids with local refinement. With proper computation of the transformation metrics and the use of TVD artificial viscosity, the MacCormack scheme is stable and is able to solve problems with sharp discontinuities.

The examples we have presented demonstrate the utility of these methods and also point out their shortcomings. Of particular concern is the lack of any error estimation near the boundaries, the poor error estimation near discontinuities, and the need to constrain the mesh to obtain any accurate error estimation. These problems must be solved in order to effectively utilize this solution scheme and error estimation procedure with an adaptive technique.

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